Structural phase transition in evolving networks

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A network as a substrate for dynamic processes may have its own dynamics. We propose a model for networks which evolve together with diffusing particles through a coupled dynamics and investigate emerging structural property. The model consists of an undirected weighted network of fixed mean degree and randomly diffusing particles of fixed density. The weight w of an edge increases by the amount of traffics through its connecting nodes or decreases by a constant factor. Edges are removed with the probability $P_{rew}=1/(1+w)$ and replaced by new ones having $w=0$ at random locations. We find that the model exhibits a structural phase transition between the homogeneous phase characterized by an exponentially decaying degree distribution and the heterogeneous phase characterized by the presence of hubs. The hubs emerge as a consequence of a positive feedback between the particle and the edge dynamics.

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Complex networks have been the subject of extensive researches for the last decade. They have a heterogeneous structure, which makes them distinct from the periodic lattice and the random network. Researches have been focused on characterizing the structure and revealing the mechanism leading to it [1–4]. Some complex networks play the role of a substrate on which dynamic processes, either equilibrium or nonequilibrium, take place. Implication of the structural heterogeneity on the dynamic processes has also attracted a lot of interests [5,6].

In most studies dynamics of a network itself and additional degrees of freedom on it are treated separately. These approaches are meaningful when characteristic time scales associated with each of them are completely separated. When they are comparable, one needs to consider the dynamics of both kinds of degrees of freedom simultaneously. Along these lines, dynamic models for a social network coupled with a game-theoretical dynamics or an opinion dynamics were studied in Refs. [7–10]. Also studied were evolving network models coupled with self-organized critical systems [11,12] and with a diffusing particle system [13].

The coupled dynamics was also studied in a dynamic model for a transportation or an information network by the present authors in Ref. [14]. This study was motivated by the synaptic plasticity in neural networks [15]. Synaptic links in a neural network may strengthen or weaken depending on synaptic activities, which results in a plastic deformation of a network. In the model [14], particles diffuse over a network and edges are rewired at the rate depending on particle flows in such a way that edges contributing more to transport are more robust. It was found that the coupled dynamics leads to an instability toward the formation of a hub. Although the model is useful in studying the dynamical origin for the emergence of a hub, it lacks a parameter with which one can control the strength of the instability.

In this paper, we consider a model as an extension of the study in Ref. [14]. The model consists of an undirected weighted network and diffusing particles. There are N nodes with the mean degree $\langle k\rangle$ and particles with the density ρ. An edge e is assigned to a weight $w_e \geq 0$, and a node can accommodate multiple particles. The edges, weights, and particles evolve in time as follows. At each time step, every particle hops independently to a neighboring node selected at random. Whenever a node is reached by a particle, the weight of all edges attached to it is increased by unity. Then, with the probability $p_{rew}=1/(1+w_e)$, each edge e is removed and replaced by a new one with $w=0$ between a pair of nodes selected randomly. As a regularization procedure, the weight of all edges are degraded by the factor r, i.e., $w_e \to (1-r)w_e$ for all e.

This model allows one to study the emerging property of a complex network evolving through a coupled dynamics with a transport system. For simplicity, we adopt the system of noninteracting random walkers as a transport system. Each edge is assigned to the weight which measures the amount of traffics handled by connecting nodes. The random walkers move along edges, while edges are rewired at the rate which is a decreasing function of the weight. That is to say, the more contribution to the traffic an edge makes, the more robust it is.

When the degradation factor r is zero, the model reduces to the one studied in Ref. [14]. It was shown that the coupled dynamics between the edges and particles leads to an instability toward the formation of a hub. Interestingly, the hub emerges in the two distinct ways. When the particle density is low, the hub appears spontaneously after overcoming a dynamic barrier via statistical fluctuations. On the other hand, when the particle density is high, nodes compete for edges and one of them survives as a hub eventually. In the original model, the weight of an edge can increase indefinitely, which may not be the case when an edge loses its strength due to, e.g., aging. The current model incorporates such an effect by introducing the degradation factor r. We will show that the model with the degradation factor exhibits richer behaviors.

The degradation factor r limits the growth of the edge weight. Hence, the model can have a stationary state with nonzero values of r. Furthermore, there may be a phase transition. When r is large, one expects that edges are rewired at so high rates that the network remains like a random network without any hub. In the opposite case with small r, edges attached to a certain node may become robust by gaining

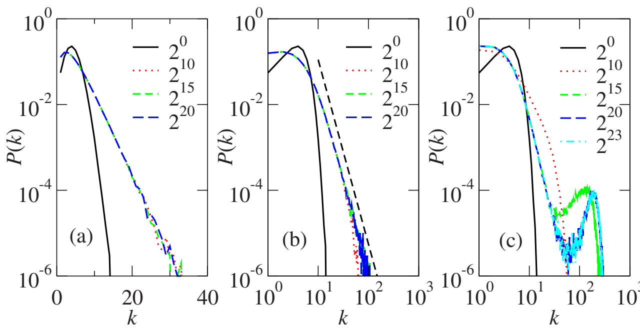

FIG. 1. (Color online) Time evolution of the degree distribution at (a) $r=0.20$, (b) 0.055, and (c) 0.01. The curves at later time steps are on top of each other since the system reaches the stationary state. The network size is $N=10^3$ and the particle density is $\rho=1.0$. Each data set is obtained by averaging over $N_S=10^3$ samples. The dashed line in (b) has a slope of −4.3.

larger weights. Such a node can grow into a hub as in the case with $r=0$. Practically, we define the hub as a node whose degree scales algebraically with the total number of nodes.

We have performed numerical Monte Carlo simulations in order to examine whether there is a transition between the states with and without hubs. We present detailed results of numerical simulation studies. Initially, we start with a random network over which particles are distributed randomly. The mean degree is fixed to $\langle k\rangle=4$. Then we study the time evolution and the stationary-state property of the system.

The degree distribution has been measured as one varies r with fixed $\rho=1$, which is presented in Fig. 1. The degree distribution function $P(k)$ is defined as the fraction of nodes having k edges. At $r=0.2$, the degree distribution in the stationary state aligns along a straight line in the semilog plot [see Fig. 1(a)]. This means that the degree distribution follows an exponential decay as $P(k)\sim e^{-k/k_0}$. We observe a distinct feature at $r=0.01$. While the degree distribution decays at small values of k, there appears a peak in the large k region. We will show later that both the number of nodes contributing to the peak and the degree of them scale algebraically with the network size N, respectively. Namely, the peak signals the emergence of multiple hubs. In the intermediate case with $r=0.055$, the degree distribution follows a power law:

$$P(k) \sim k^{-\gamma}, \tag{1}$$

with the exponent $\gamma\simeq 4.3$.

Numerical data presented in Fig. 1 show that the model undergoes a structural phase transition between the stationary states with and without hubs. Such phases will be denoted as a *heterogeneous* phase and a *homogeneous* phase, respectively. The transition point can be estimated accurately from the effective exponent defined as

$$\gamma_{eff}(k) = -\ln[P(ak)/P(k)]/\ln a, \tag{2}$$

with a constant $a=2$. As a function of k, it will grow without bound if the degree distribution decays exponentially. In the presence of the peak for hubs, the effective exponent will be

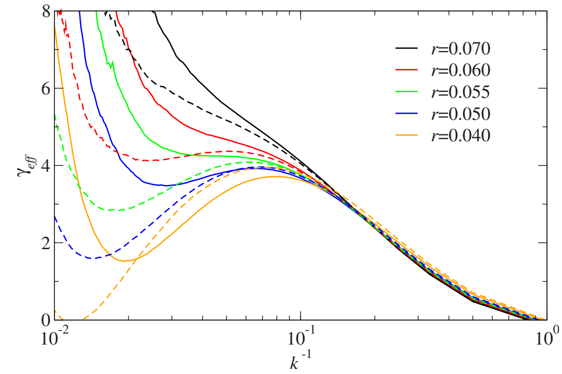

FIG. 2. (Color online) Plots of the effective exponent $\gamma_{eff}(k)$ for the degree distribution with the parameter values $r=0.070$, 0.060, 0.055, 0.050, and 0.040 from top to bottom. The network size of $N=1000$ (solid lines) and $N=2000$ (dashed lines) and $\rho=1.0$.

a nonmonotonic function of k. If the degree distribution follows asymptotically a power law as $P(k)\sim k^{-\gamma}$, the effective exponent will converge to γ.

In Fig. 2, we present the plot of the effective exponent at several values of r at $\rho=1.0$. When $N=1000$ and $r=0.055$, there appears a plateau at $\gamma_{eff}\simeq 4.3$. Above and below $r=0.055$, the effective exponent plot shows the characteristic of the exponential decay and the peak for hubs, respectively. Hence, we conclude that the structural phase transition takes place at $r=r_c=0.055(5)$. The degree distribution at $r=r_c$ follows the power-law decay with the exponent $\gamma\simeq 4.3$. Note that the plateau region widens as N increases. This implies that the blowup of γ_{eff} for $k^{-1}\lesssim 0.02$ is due to a finite-size effect. Repeating this analysis at other values of r and N, we obtain the numerical phase diagram as shown in Fig. 3. Although a finite-size effect is rather large up to $N=2000$, the numerical phase diagram shows a clear evidence for the phase transition. The degree exponent remains almost constant along the phase boundary.

We present an analytic theory, which explains the mechanism of the phase transition. In order to describe the dynamics, one needs to consider the degrees of all nodes, the weights of all edges, and the particle occupation number at all nodes. It is difficult to consider the whole dynamics, so we develop an approximate theory as below.

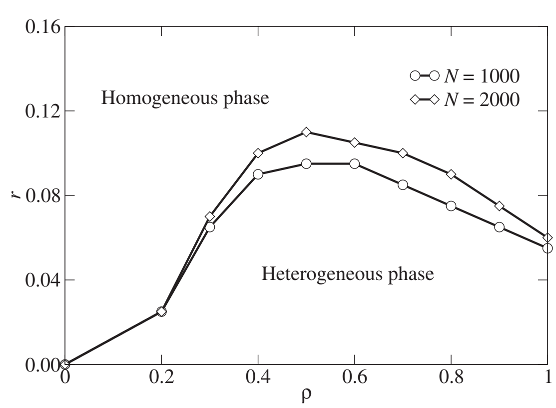

FIG. 3. Numerical phase diagram obtained from simulations with $N=1000$ and 2000.

Consider an arbitrary node i_0. The degree of the node at time t will be denoted by $K(t)$. We assume that there exists a characteristic value of the weight $\Omega(t)$ for the K edges of i_0. We make a further assumption that the other part can be regarded as a uniform medium, where edges are rewired at a constant rate s. For these assumptions, our description is a mean-field theory, which works only when structural heterogeneity of the system is negligible. It is not valid in the heterogeneous phase with hubs since it loses self-consistency. Nevertheless, we can learn when and why the structural phase transition will occur from the breakdown of self-consistency.

An edge gains a weight when particles arrive at its connecting nodes. Diffusing particles on complex networks reach the stationary state very rapidly [16,17]. So we adopt a quasistationary-state assumption that the particle distribution is approximated by the stationary-state distribution to a given network at each moment. The stationary-state particle distribution function is strictly proportional to the degree [16]. The quasistationary-state assumption, which was also made in Ref. [14], allows us to integrate out the particle degrees of freedom. Then, the rate equation for the weight variable Ω in time-continuum limit is given by

$$\frac{d\Omega}{dt} = \frac{\rho K}{\langle k \rangle} + \rho - r\Omega. \tag{3}$$

The first two terms account for the gain coming from the visit of particles to the node i_0 and its neighboring node, respectively. The last term accounts for the loss due to the degradation.

The degree variable $K(t)$ follows the rate equation:

$$\frac{dK}{dt} = s - \frac{K}{1+\Omega}. \tag{4}$$

The first term accounts for the attachment of a randomly rewired edge to i_0, and the second term accounts for the rewiring of each of the K edges with the probability $1/(1+\Omega)$.

The flow governed by Eqs. (3) and (4) has an attracting fixed point when $r > r_c$ with

$$r_c = \frac{s\rho}{\langle k \rangle}. \tag{5}$$

The fixed point is located at

$$K_0 = \frac{s(r+\rho)}{(r-r_c)}, \quad \Omega_0 = \frac{(\rho + r_c)}{(r-r_c)}. \tag{6}$$

Irrespective of an initial condition, the flow converges to the fixed point.

When $r < r_c$, there does not exist an attracting fixed point at finite values of K and Ω. They grow unboundedly. The blowup solution invalidates the assumption that the network remains homogeneous. It signals the emergence of a hub. The flow pattern is sketched schematically in Fig. 4.

The mean-field theory confirms that the structural phase transition indeed takes place. It also reveals the mechanism for the emergence of the hub. Following Eq. (3), an increase in K accelerates the growth of Ω. Likewise, an increase in Ω accelerates the growth of K. This shows that there is a positive feedback between K and Ω. Actually, the edge weight growth is driven by diffusing particles. Therefore, we conclude that the coupled dynamics of the network and diffusing particles can lead to the heterogeneous network structure.

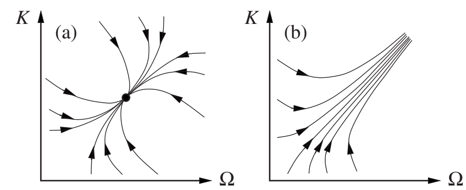

FIG. 4. Schematic flow diagram in the (Ω, K) plane when (a) $r < r_c$ and (b) $r > r_c$. The fixed point (Ω_0, K_0) is represented with a filled circle in (a).

Under the quasistationary-state assumption that the number of particles on a node is strictly proportional to its degree, our model looks similar to the one studied in Ref. [18]. The latter model, where edges are rewired preferentially to higher degree nodes, also displays a structural phase transition accompanied by the condensation of edges. The difference is that our model does not assume explicitly the preferential rewiring. Instead, it is generated dynamically. Furthermore, the edge weight and the rewiring probability depend not only on the current values of the degree and the particle number but also on their time history as one can see in Eq. (3).

We add a few remarks on the phase diagram. The numerical phase diagram in Fig. 3 shows a re-entrant behavior from the homogeneous phase through the heterogeneous phase to the homogeneous phase again as one increases the value of ρ. The re-entrance is allowed by Eq. (5) since the rewiring rate s, the number of rewrired edges per unit time divided by the total number of nodes, can depend on r, ρ, and $\langle k \rangle$. Our mean-field theory does not predict the function form of s. We could measure its value only numerically. We found that $s\rho$ increases from 0.25 to 0.30 as ρ varies from 0.2 to 0.3 at fixed $r=0.2$ and $\langle k \rangle = 4$. On the other hand, $s\rho$ decreases from 0.21 to 0.19 as ρ varies from 0.8 to 1.0 at fixed $r=0.1$ and $\langle k \rangle = 4$. This tendency shows that Eq. (5) is consistent with the re-entrant phase diagram qualitatively. Note also that s is an increasing function of $\langle k \rangle$. The more edges there are, the more edges are rewired. This indicates that r_c does not necessarily decreases as one increases $\langle k \rangle$. Numerically, we found that $r_c(\rho=1.0, \langle k \rangle = 8) = 0.070(5)$, which is even larger than $r_c(\rho=1.0, \langle k \rangle = 4) = 0.055(5)$.

The model displays an interesting scaling behavior in the heterogeneous phase. Figure 5(a) shows the degree distribution at several values of N to a given value of $\rho=1.0$ and $r=0.01$ belonging to the heterogeneous phase. There is a peak corresponding to the hubs. As N increases, the peak shifts to the right but does not sharpen nor broaden. This suggests that the number of hubs scales algebraically with N and that their degrees have the same order of magnitude scaling algebraically with N. One can measure the number of hubs N_{hub} from the spectral weight of the peak in $P(k)$. The numerical data for N_{hub} are plotted in Fig. 5(b), which shows that it follows a power law:

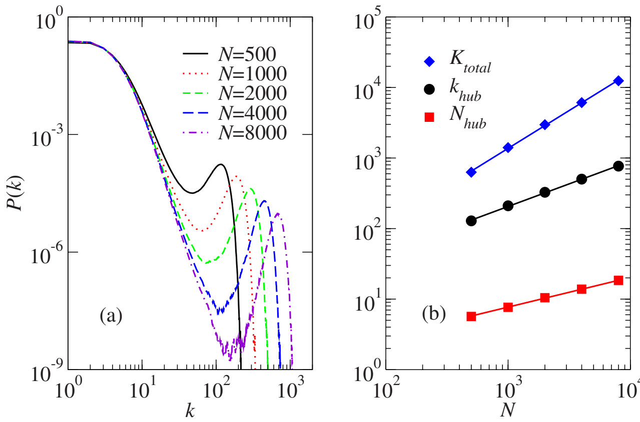

FIG. 5. (Color online) Degree distribution in (a) and k_{hub}, N_{hub}, and K_{total} in (b) in the stationary state in systems with $\rho=1.0$ and $r=0.01$. The straight lines in (b) are guides for the eyes.

$$N_{hub} \sim N^{q}, \tag{7}$$

with $q \simeq 0.43$. Figure 5(b) also shows that the maximum degree k_{max} among all nodes follows a power law:

$$k_{max} \sim N^{\theta}, \tag{8}$$

with $\theta \simeq 0.64$. The sum of the exponents is close to unity, which indicates that the total number of edges K_{total} attached to the hubs is proportional to N. Numerical data in Fig. 5(b) show that

$$K_{total} \sim N^{\mu}, \tag{9}$$

with $\mu \simeq 1.07$ which is very close to 1. The exponents q, θ, and μ remain constant in heterogeneous phase up to statistical errors.

The structural phase transition has a similarity to a condensation transition in the zero range process (ZRP) [19]. In the ZRP, a unit mass hops from one site to another on a given graph with a hopping rate depending on the total masses on a departing site. The masses may undergo a condensation transition between a fluid phase and a condensed phase. In the fluid phase, masses are distributed uniformly. When there is a strong on-site attraction among masses, a quenched disorder, or a structural heterogeneity in an underlying graph, a finite fraction of the total masses can condense on a single site to form a macroscopic condensate [17,19–21].

The similarity between the condensation in the ZRP and the emergence of hubs in network dynamics have already been noticed in Ref. [22]. Regarding nodes and edges in the network dynamics as sites and masses in the ZRP, respectively, a hub can be seen as a condensate of edges. Apart from the similarity, the structural phase transition in our model has a distinct feature. In the context of the condensation, there exist multiple number of condensates ($N_{hub} \sim N^{q>0}$) and the condensates are mesoscopic, that is to say, their size scales sublinearly in N ($k_{hub} \sim N^{\theta<1}$). This is contrasted to the ZRP with a local dynamics, which has a single macroscopic condensate in the condensed phase [19,20].

Recent studies report that multiple mesoscopic condensates appear when the dynamics is nonlocal in the sense that the mass hopping rate depends not only on the local occupation number but also on the global parameter such as the system size [22,23]. In our model, the edge rewiring dynamics is purely local but coupled with the weight dynamics. We leave it as a future work to understand the origin for the emergence of multiple mesoscopic condensates in the context of the ZRP.

In summary, we propose a dynamic model for networks with the edge rewiring dynamics which is coupled to the particle diffusion dynamics. The model displays a structural phase transition between the homogeneous phase and the inhomogeneous phase. The former is characterized by an exponential degree distribution and the latter is characterized by the multiple mesoscopic hubs. Those hubs are the consequence of the positive feedback between the edge and particle dynamics. Our work uncovers a mechanism how structural heterogeneity of complex networks can emerge.

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